JOURNAL OF MULTIVARIATE ANALYSIS 25, 1–9 (1988)

On Multivariate Mean Remaining Life Functions

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Communicated by P. R. Krishnaiah

The concept of the mean remaining life function is extended to the multivariate case. Some general characterization properties of multivariate mean remaning life functions (mmrl) are studied, it is shown that they determine the joint pdf uniquely. Inter alia a basic convergence result is proved for a sequence of mmrl functions. The multivariate loss of memory property of a pdf is characterized in terms of the mmrl function. The relationship between the mmrl function and the hazard-gradient is discussed. © 1988 Academic Press, Inc.

1. INTRODUCTION

In recent years considerable attention has been paid to the problem of characterizing the probability distribution function (pdf) F of a random variable (r.v.) X based on conditional expectation in general, and in particular on its mean remaining life (mrl) function. See, for example, Shanbhag [26], Hamdan [16], Kotlarski [18], Funk [10, 11], Swartz [27], Dallas [9], Laurent [20], Arnold [1], Gupta [15], Nagaraja [24], Kotz and Shanbhag [19], and Bhattacharjee [5]. It is natural to ask if there is an analog in the multivariate case. As far as we know there have been two instances where multivariate extensions of the univariate mrl

Received August 1982; revised June 29, 1984.

AMS 1980 subject classifications: Primary G2H05; Secondary 62E10.

Key words and phrases: Conditional expectation, multivariate mean remaining life function, multivariate distributions, stability theorem, loss of memory property, multivariate exponential distributions.

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0047-259X/88 $3.00

function have been implicitly defined. The most general form proposed by Buchanan and Singpurwalla in [8] is

$$g(\mathbf{x}) = \int_{\mathbf{0}}^{\infty} P(\mathbf{X} > \mathbf{x} + \mathbf{t})\, d\mathbf{t} / P(\mathbf{X} > \mathbf{x}), \qquad \mathbf{x} \geqslant \mathbf{0}; \tag{1}$$

(their others are special cases of (1)). Ghosh and Ebrahimi [14] also made use of $g(\mathbf{x})$ as defined in (1). Although $g(\mathbf{x})$ seems a reasonable, direct extension of the univariate mrl function $\gamma^*(x)$,

$$\gamma^*(x) = E(X - x \mid X > x) = \int_x^{\infty} P(X > t)\, dt / P(X > x), \qquad P(X > x) > 0;$$

nevertheless, it does not have the most essential property of the univariate mrl function. That is, it does not determine the corresponding pdf uniquely.

In the present note we supply a definition of a mean remaining life function that does characterize the corresponding pdf uniquely.

2. A Multivariate Mean Remaining Life Function

Let F be the joint pdf of a random vector $\mathbf{X}$ on R^p ($p \geqslant 1$). The extended real valued vector $\mathbf{b} = (b_1, b_2, ..., b_p)$ is called the supremum of the support (sos) of F if

$$b_i = \inf\{x \colon F_{X_i}(x) = 1\}, \tag{2}$$

where F_{X_i} is the ith univariate marginal distribution of F, and infimum of an empty set is defined to be ∞.

2.1. Definition. Let $\mathbf{X} = (X_1, X_2, ..., X_p)$ be a random-vector on R^p with joint pdf F. Let $\mathbf{b} = (b_1, b_2, ..., b_p)$ denote the sos of F, $X_i^+ = \max(0, X_i)$ and suppose

$$E(X_i^+) < \infty \qquad \text{for} \quad i = 1, 2, ..., p. \tag{3}$$

Define a vector-valued Borel measurable function $\boldsymbol{\gamma}(\mathbf{t})$ on R^p by

$$\boldsymbol{\gamma}(\mathbf{t}) = (\gamma_1(\mathbf{t}), \gamma_2(\mathbf{t}), ..., \gamma_p(\mathbf{t})) = E(\mathbf{X} - \mathbf{t} \mid \mathbf{X} \geqslant \mathbf{t}), \tag{4}$$

for all $\mathbf{t} \in R^p$, $\mathbf{t} < \mathbf{b}$ such that $P(\mathbf{X} \geqslant \mathbf{t}) > 0$. The vector-valued function $\boldsymbol{\gamma}(\mathbf{t})$ will be called the *multivariate mean remaining life function* (mmrl).

On the set $\{\mathbf{t} \colon P(\mathbf{X} \geqslant \mathbf{t}) = 0\}$, the function $\boldsymbol{\gamma}(\mathbf{t})$ can be defined in an arbitrary fashion.

The following notational convention will be used repeatedly. For any scaler w and any vector $\mathbf{t} \in R^{p-1}$ the symbol $(w_{(i)}, \mathbf{t})$ will denote the vector $(t_1, ..., t_{i-1}, w, t_{i+1}, ..., t)$.

Suppose $\mathbf{x}$ has sos $\mathbf{b}$ and mmrl $\boldsymbol{\gamma}(\mathbf{x})$. Let A_i denote the set $\{z: \lim_{x \uparrow z}(\sup_{\mathbf{t}} \gamma_i(x_{(i)}, \mathbf{t}))$ exists and equals $0\}$, where $\mathbf{t} \in R^{p-1}$. Then for $i = 1, 2, ..., p$,

$$b_i = \begin{cases} \inf\{z: z \in A_i\} & \text{if } A_i \neq \varnothing \\ \infty & \text{if } A_i = \varnothing. \end{cases} \tag{5}$$

The proof parallels that provided by Kotz and Shanbhag [19] in the univariate case.

A convenient representation for the mmrl is provided by, for $i = 1, 2, ..., p$,

$$\gamma_i(\mathbf{t}) = E[X_i - t_i \mid \mathbf{X} \geqslant \mathbf{t}] = \frac{1}{\bar{F}(\mathbf{t})} \int_{t_i}^{\infty} \bar{F}(x +_{(i)}, \mathbf{t})\, dx, \tag{6}$$

where $\mathbf{t} < \mathbf{b}$ and

$$\bar{F}(\mathbf{t}) \overset{\text{def}}{=} P(X_1 \geqslant t_1, X_2 \geqslant t_2, ..., X_p \geqslant t_p). \tag{7}$$

The proof parallels that given by Kotz and Shanbhag in the univariate case (note that we condition on $\mathbf{X} \geqslant \mathbf{t}$ rather than $X_i > t_i$). Equation (6) shows that $\gamma_i(\mathbf{t})$ is the ratio of two left continuous functions and hence itself is left continuous. Clearly $\mathbf{t}$ is a point of continuity of $\bar{F}$ if and only if it is a point of continuity of γ_i $(i = 1, 2, ..., p)$. It is evident from (6) that knowledge of the mmrl of $\mathbf{X}$ is sufficient to determine the mmrl of any subcollection of the x_i's (by letting superfluous t_i's converge to $-\infty$).

Conversely we can express $\bar{F}$ as a function of $\boldsymbol{\gamma}$ as follows. For any $\mathbf{s} < \mathbf{t} < \mathbf{b}$

$$\frac{\bar{F}(\mathbf{t})}{\bar{F}(\mathbf{s})} = \prod_{i=1}^{p} \left[\frac{\gamma_i(t_1, ..., t_{i-1}, s_i, s_{i+1}, ..., s_p)}{\gamma_i(t_1, ..., t_{i-1}, t_i, s_{i+1}, ..., s_p)} \exp \left(-\int_{s_i}^{t_i} \frac{du}{\gamma_i(t_1, ..., t_{i-1}, u, s_{i+1}, ..., s_p)} \right) \right] \tag{8}$$

and

$$\bar{F}(\mathbf{t}) = \lim_{\mathbf{s} \to -\boldsymbol{\infty}} \prod_{i=1}^{p} \left[\frac{\gamma_i(t_1, ..., t_{i-1}, s_i, s_{i+1}, ..., s_p)}{\gamma_i(t_1, ..., t_{i-1}, t_i, s_{i+1}, ..., s_p)} \exp \left(-\int_{s_i}^{t_i} \frac{du}{\gamma_i(t_1, ..., t_{i-1}, u, s_{i+1}, ..., s_p)} \right) \right] \tag{9}$$

where $-\boldsymbol{\infty} = (-\infty, ..., -\infty)'$.

Note that (8) is a consequence of the fact that for any i,

$$
\begin{aligned}
&-\int_{s_i}^{t_i} \frac{du}{\gamma_i(t_1, ..., t_{i-1}, u, s_{i+1}, ..., s_p)} \\
&\quad = \log\left[\frac{\gamma_i(t_1, ..., t_i, s_{i+1}, ..., s_p)\,\bar{F}(t_1, ..., t_i, s_{i+1}, ..., s_p)}{\gamma_i(t_1, ..., t_{i-1}, s_i, ..., s_p)\,F(t_1, ..., t_{i-1}, s_i, ..., s_p)}\right], \\
&\qquad i = 1, 2, ..., p.
\end{aligned}
$$

2.2. *Remarks.* 1. From (5), (6), (8), and (9) it follows that knowing the mmrl function $\boldsymbol{\gamma}(\mathbf{t})$ for all $\mathbf{t}$ $(-\infty\mathbf{e} < \mathbf{t} < \mathbf{b})$ determines the corresponding pdf F uniquely, and, vice versa. This is consistent with the univariate case. If in addition we assume that $\bar{F}$ has a continuous gradient $\nabla\bar{F} = (\partial\bar{F}/\partial x_1, ..., \partial\bar{F}/\partial x_p)$, then it is not difficult to check that Eq. (8) can be written as:

$$
\frac{\bar{F}(\mathbf{t})}{\bar{F}(\mathbf{s})} = \exp\left[-\int_{\mathbf{s}}^{\mathbf{t}} \left(\frac{\gamma_1'(\mathbf{u})+1}{\gamma_1(\mathbf{u})}, ..., \frac{\gamma_p'(\mathbf{u})+1}{\gamma_p(\mathbf{u})}\right)\cdot d\mathbf{u}\right], \tag{10}
$$

where

$$
\int_{\mathbf{s}}^{\mathbf{t}} \left(\frac{\gamma_1'(\mathbf{u})+1}{\gamma_1(\mathbf{u})}, ..., \frac{\gamma_r'(\mathbf{u})+1}{\gamma_p(\mathbf{u})}\right)\cdot d\mathbf{u}
$$

denotes a line integral, and is independent of the path (piecewise smooth), and $\gamma_i'(\mathbf{u}) =^{\text{def}} (\partial/\partial u_i)\,\gamma_i(\mathbf{u})$.

2. $\mathbf{X}$ has independent marginals if and only if

$$
\gamma_i(\mathbf{t}) = \gamma_{X_i}(t_i) \qquad \forall i, \forall \mathbf{t} < \mathbf{b}.
$$

A stability result, analogous to the uninvariate case is given by 2.3.

2.3. THEOREM. *Let $\{F_n : n = 1, 2, ...\}$ be a sequence of pdf's on R^p, with a corresponding sequence of mmrl functions $\{\gamma^n : n = 1, 2, ...\}$, respectively. Let F be a pdf on R^p with corresponding mmrl functions γ and supremum of support* $\mathbf{b}$. *Suppose that these two conditions hold*:

(i) $\limsup_{n\to\infty} \int_{\|\mathbf{x}\| > T} dF_n(\mathbf{x}) \to 0$ *as* $T \to \infty$ (*tightness condition*)

(ii) $\bar{F}_{X_i^n}(x) \leqslant g_i(x)$ $(i = 1, 2, ..., p)$, $\forall n \geqslant 1$ *and* $\forall x \geqslant 0$,

where g_i's are borel measurable functions, integrable with respect to Lebesgue measure on R^1; $\bar{F}_{X_i^n}(x) = 1 - F_{X_i^n}(x-)$, where $F_{X_i^n}$ denotes the pdf of the ith *univariate marginal of* F_n, $i = 1, 2, ..., p$. *Let* C *be the set of continuity points* $\mathbf{x} < \mathbf{b}$ *of* F *(and consequently of* γ*). Then*

$$\gamma^n(\mathbf{x}) \to \gamma(\mathbf{x}) \quad \text{as} \quad n \to \infty, \quad \forall \mathbf{x} \in C$$

(*i.e.*, $\gamma_i^n(\mathbf{x}) \to \gamma_i(\mathbf{x})$, $i = 1, 2, ..., p$) *if and only if*

$$F_n(\mathbf{x}) \to F(\mathbf{x}) \quad \text{as} \quad n \to \infty, \quad \forall \mathbf{x} \in C,$$

i.e., $F_n \to^w F$.

The proof is a straightforward generalization of the univariate proof. It is not difficult to construct examples to show that the convergence theorem (2.3) need not be true if either condition (i) or (ii) is relaxed. For instance, consider the two examples of Kotz and Shanbhag [19].

Note. An alternative candidate mmrl function is $\gamma^*(\mathbf{x}) = E(\mathbf{x} - \mathbf{t} \mid \mathbf{X} > \mathbf{t})$. If F is continuous then $\gamma \equiv \gamma^*$. In general $\gamma(t_{1+}, ..., t_{p+}) = \gamma^*(t_1, ..., t_p)$ so that γ^* determines the corresponding pdf uniquely as does γ. Almost all our results for γ remain valid for γ^* if we replace $\bar{F}(x)$ by $P(\mathbf{X} > \mathbf{x})$ whenever necessary.

3. Relationship between the MMRL Function and the Hazard-Gradient

In the context of reliability theory, in analyzing multivariate survival data from an absolutely continuous pdf F on P_+^p, the hazard-gradient plays a major role.

3.1. Definition. Let $\mathbf{X} = (X_1, ..., X_p)$ be a random-vector on R_+^p with the joint survival function S,

$$S(x_1, ..., x_p) = P(X_1 > x_1, ..., X_p > x_p),$$

where $\mathbf{x} = (x_1, ..., x_p)$ is in R_+^p. Let $R(\mathbf{x}) = -\log S(\mathbf{x})$ for $\mathbf{x}$ in the set $\{\mathbf{x}: S(\mathbf{x}) > 0\}$. The function $R(\mathbf{x})$ is called the *hazard function* (analogous to the univariate case). *If* R has a gradient $\mathbf{r} = \nabla R$, then $\mathbf{r}$ is called the *multivariate hazard rate* or *vector multivariate hazard rate* or *hazard-gradient*. More explicitly, if for $\mathbf{x} \in \{\mathbf{x}: S(\mathbf{x}) > 0\}$,

$$r_i(\mathbf{x}) = \frac{\partial}{\partial x_i} R(\mathbf{x}), \qquad i = 1, 2, ..., p,$$

Then $\mathbf{r}(\mathbf{x}) = (r_1(\mathbf{x}), ..., r_p(\mathbf{x}))$ is the hazard-gradient. The hazard-gradient has been discussed by Block [6, 7], Johnson and Kotz [17], Marshall [21, 22], and Barlow and Proschan [3, 4].

If F is a pdf on R_+^p with sos $\mathbf{b}$ and such that $\nabla\bar{F}$ exists. Then the corresponding mmrl $\boldsymbol{\gamma}$ and hazard-gradient $\mathbf{r}$ are related by

$$r_i(\mathbf{t}) = \frac{(\partial/\partial t_i)\,\gamma(\mathbf{t}) + 1}{\gamma_i(\mathbf{t})}, \qquad i = 1, 2, ..., p, \quad \mathbf{t} < \mathbf{b}. \tag{11}$$

This follows, using the continuity and differentiability of $\bar{F}$, from differentiating the expression for $\gamma_i(\mathbf{t})$ with respect to t_i and noting that

$$r_i(\mathbf{t}) = \frac{\partial}{\partial t_i}\bar{F}(\mathbf{t})/\bar{F}(\mathbf{t}), \qquad i = 1, ..., p.$$

Expression (11) and our earlier results can be used to rederive Marshall's [21] assertion that the hazard gradient uniquely determines the pdf.

4. A Characterization of the Multivariate Loss of Memory Property

We begin this section with the following definition:

4.1. Definition. A pdf F on R^p is said to have *multivariate loss of memory property* (MLMP) if

$$P(X_1 > x_1 + \Delta, ..., X_p > x_p + \Delta \mid X_1 > x_1, ..., X_p > x_p) = P(X_1 > \Delta, ..., X_p > \Delta),$$

or equivalently,

$$\bar{F}(x_1 + \Delta +, ..., x_p + \Delta +) = \bar{F}(x_1 +, ..., x_p +)\,\bar{F}(\Delta +, ..., \Delta +). \tag{12}$$

for all $x_1, ..., x_p$ and Δ. When $p = 1$ $(p = 2)$ we use LMP (BLMP) instead of MLMP.

If F has the MLMP it is not difficult to verify that $\bar{F}(0+, ..., 0+) = 1$ and that $\bar{F}$ is continuous in each variable. One can thus rewrite (12) as

$$\bar{F}(x_1 + \Delta, x_2 + \Delta, ..., x_p + \Delta) = \bar{F}(x_1, x_2, ...\, x_p)\bar{F}(\Delta, \Delta, ...\, \Delta)$$
$$\forall x_1, x_2, ...x_p, \qquad \Delta \geqq 0,$$

A very simple characterization of the MLMP is available in terms of the mmrl function. That is, for F continuous on R_+^p, F has the MLMP if and only if

$$\gamma_i(\mathbf{t}+\Delta\cdot\mathbf{1})=\gamma_i(\mathbf{t}), \quad \forall \mathbf{t}\in R_+^p \quad \text{and} \quad \forall \Delta \geqslant 0 \quad \text{for} \quad i=1,2,...,p. \tag{13}$$

The proof of this assertion follows from (8) and (10) (with $\mathbf{s}=\mathbf{0}$) using the definition of MLMP.

Remarks. (i) $\mathbf{X}=(X_1,...,X_p)$ be a random vector on R_+^p with joint pdf F. If $\mathbf{X}$ (or, equivalently, F) has the MLMP, then immediately it follows that $X^*=\min_{1\leqq i\leqq p} X_i$ also has the LMP property, and consequently X^* is exponentially distributed.

(ii) Let F be the pdf of a random vector $\mathbf{X}$ on R_+^p with the MLMP, and corresponding mmrl function $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,...,\gamma_p)$. Obviously $\gamma_i(t\cdot\mathbf{1})=\gamma_i(0\cdot\mathbf{1})=C_i$, a positive constant, $\forall t\geqslant 0$, where $C_i=E(X_i)$, $i=1,2,...,p$. Nevertheless the condition $\gamma_i(t\cdot\mathbf{1})=C_i$, $\forall t\geqslant 0$ and for $i=1,...,p$ neither implies nor is implied from the condition that $X^*=\min_{1\leqq i\leqq p} X_i$ is exponentially distributed. For example, consider

$$\bar{F}_{\substack{(x_1,x_2)\\(X_1,X_2)}}=\tfrac{1}{2}\exp(-x_1-x_2)+\tfrac{1}{2}\exp-\max(x_1,x_2)\,x_1,x_2\geqslant 0.$$

and

$$\bar{F}_{\substack{(x_1,x_2)\\(X_1,X_2)}}=\exp[-(x_1^2+x_2^2)^{1/2}],\,x_1,x_2\geqslant 0.$$

5. An Example

Consider the sdf F, given by

$$\bar{F}(x_1,x_2)=\begin{cases}\lambda_1+\lambda_2-\lambda_2')^{-1}\{\lambda_1\exp[-(\lambda_1+\lambda_2-\lambda_2')\,x_1-(\lambda_0+\lambda_2')\,x_2] \\ \qquad +\,(\lambda_2-\lambda_2')\exp(-\lambda x_2)\}, \qquad x_1\leqq x_2, \\ (\lambda_1+\lambda_2-\lambda_1')^{-1}\{\lambda_2\exp[-(\lambda_0+\lambda_1')\,x_1-(\lambda_1+\lambda_2-\lambda_1')\,x_2] \\ \qquad +\,(\lambda_1-\lambda_1')\exp(-\lambda x_1)\}, \qquad x_1\geqq x_2,\end{cases} \tag{14}$$

where the parameters $\lambda_0,\lambda_1,\lambda_2,\lambda_1',\lambda_2'$ are all positive, $\lambda=\lambda_0+\lambda_1+\lambda_2$, $x_1,x_2>0$. This distribution, which generalizes both the bivariate exponential distribution (BVE) of Marshall and Olkin [23] and the BVE of Freund [12] was introduced by Proschan and Sullo [25] as an example for some sampling and inference results. The above survival function can be found in Friday [13], where the distribution is referred to as the PSE (Proschan–Sullo extension) distribution. The first component of the mmrl function γ corresponding to PSE distribution is given by

$$\gamma_1(\mathbf{t}) = \begin{cases} \dfrac{\lambda_1\{\exp[\theta_2(t_2-t_1)]-1\}+\theta_2(\lambda_2-\lambda_2')(t_2-t_1) \\ \quad +\theta_2^2[\lambda_2+\theta_1(\lambda_1-\lambda_1')(\lambda_0+\lambda_1')][\theta_1(\lambda_0+\lambda_1')]^{-1}}{\theta_2\{\lambda_1\exp[\theta_2(t_2-t_1)]+\lambda_2-\lambda_2'\}}, & t_1 \leqslant t_2, \\ \dfrac{\lambda_2(\lambda_0+\lambda_1')\exp[\theta_1(t_1-t_2)]+(\lambda_1-\lambda_1')(\lambda_0+\lambda_1+\lambda_2)^{-1}}{\lambda_2\exp[\theta_1(t_1-t_2)]+\lambda_1-\lambda_1'}, & t_1 \geqslant t_2, \end{cases} \tag{15}$$

where $\theta_1 = \lambda_1+\lambda_2-\lambda_1'$ and $\theta_2 = \lambda_1+\lambda_2-\lambda_2'$.

The expression for $\gamma_2(\mathbf{t})$ is obtained by interchanging the roles of the subscripts 1 and 2 in (15). From these expressions it is evident that the PSE has the BLMP. The marginal mean remaining life functions $\gamma_{X_1}(t)=\gamma_1(t, 0)$ and $\gamma_{X_2}(t)=\gamma_2(0, t)$ are clearly not constant so that the distribution given by (14) does not have exponential marginals. For further examples see Arnold and Zahedi [2].

ACKNOWLEDGMENT

The authors thank the referee for helpful suggestions.

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